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# Some defective secant varieties to osculating varieties of Veronese surfaces

A. Bernardi, M. V. Catalisano \*

**Abstract.** We consider the  $k$ -osculating varieties  $O_{k,d}$  to the Veronese  $d$ -uple embeddings of  $\mathbb{P}^2$ . By studying the Hilbert function of certain zero-dimensional schemes  $Y \subset \mathbb{P}^2$ , we find the dimension of  $O_{k,d}^s$ , the  $(s-1)^{th}$  secant varieties of  $O_{k,d}$ , for  $3 \leq s \leq 6$  and  $s = 9$ , and we determine whether those secant varieties are defective or not.

## 0. Introduction.

The problem of determining the dimension of the *higher secant varieties* of a projective variety is a classical subject of study. In the present paper we are concerned with the  $(s-1)^{th}$  higher secant varieties of  $O_{k,V_{n,d}}$ , where  $O_{k,V_{n,d}}$  is the  $k$ -osculating variety to the Veronese embedding  $V_{n,d}$  of  $\mathbb{P}^n$  into  $\mathbb{P}^N$  ( $N = \binom{d+n}{n} - 1$ ) via the complete linear system  $R_d$ , where  $R = K[x_0, \dots, x_n]$ , and  $K$  is an algebraically closed field of characteristic zero.

This matter has been dealt with by several authors in the last few years (see [2], [3], [4], [5], [7]). We wish to mention E. Ballico and C. Fontanari. In [2] and [3] they study the higher secant varieties of  $O_{k,V_{n,d}}$  for  $n = 2$  and  $k = 1, 2$ , and they prove the following results:

**0.1. Proposition.** *For  $k = 1$ , the  $(s-1)^{th}$  higher secant variety of the tangential variety to  $V_{2,d}$  has the expected dimension, unless  $s = 2$  and  $d = 3$ .*

**0.2. Proposition.** *For  $k = 2$ , the  $(s-1)^{th}$  higher secant variety of 2-osculating variety to  $V_{2,d}$  has the expected dimension, unless  $s = 2$  and  $d = 4$ .*

In this note, for  $n = 2$ , for  $3 \leq s \leq 6$  and  $s = 9$ , and for all  $k$ , we will determine the dimension of  $O_{k,V_{2,d}}^s$ . The methods for proving our results are similar to the ones used by Ballico and Fontanari. The basic idea is to use Terracini's Lemma (see [13]), then, via apolarity, the calculation of  $\dim O_{k,V_{2,d}}^s$  is related to the evaluation of the Hilbert function of certain 0-dimensional schemes  $Y \subset \mathbb{P}^2$  supported at  $s$  generic points (see [5]), and this is done using geometric constructions, Bezout's theorem, and the Horace Method ([11]).

In the first section we fix some notation, and describe the relationship between the higher secant varieties we want to study, and the 0-dimensional schemes  $Y$ .

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In the second section we relate the Hilbert function of  $Y$  to the Hilbert function  $H(X, d)$  of a scheme  $X$  of  $s$  generic  $(k+1)$ -fat points (Lemma 2.2, Proposition 2.6 to 2.9, and Proposition 2.12), and in Theorem 2.13 we prove the main result of this paper, i.e., for  $s \leq 6$  and  $s = 9$ :

$$\dim O_{k, V_{2,d}}^s = \min\{H(X, d) + 2s, N + 1\} - 1,$$

except when  $s = 2$ ,  $d = k + 2$ . In this case

$$\dim O_{k, V_{2,k+2}}^2 = H(T, d) - 1 = N - 1,$$

where  $H(T, d)$  is the Hilbert function of a scheme  $T$  of  $s$  generic  $(k+2)$ -fat points.

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## 1. Preliminaries and notation.

**1.1. Definition.** If  $V \subset \mathbb{P}^N$  is an irreducible projective variety, an  $m$ -fat point on  $V$  is the  $(m-1)^{th}$  infinitesimal neighborhood of a smooth point  $P$  in  $V$ , and it will be denoted by  $mP$  (i.e., the scheme  $mP$  is defined by the ideal sheaf  $\mathcal{I}_{P,V}^m \subset \mathcal{O}_V$ ). If  $\dim V = n$ , then,  $mP$  is a 0-dimensional scheme of length  $\binom{m-1+n}{n}$ . If  $X$  is the union of the  $(m-1)^{th}$  infinitesimal neighborhoods in  $V$  of  $s$  generic smooth points of  $V$ , we will say for short that  $X$  is union of  $s$  generic  $m$ -fat points on  $V$ .

**1.2. Remark.** In general it is a hard problem to determine the postulation for a union of  $m$ -fat points. If  $V = \mathbb{P}^2$ , there is a conjecture for the postulation of a generic union  $X \subset \mathbb{P}^2$  of  $s$   $m$ -fat points (e.g. see [10]): for  $s \geq 10$  the conjecture says that  $X$  is regular in any degree  $d$ . This has been proved for  $m \leq 20$  in [8], and, when  $s$  is a square, by L.Evain in [9]. For  $s \leq 9$  all the defective cases are known (e.g., see [8] or [10]), more precisely, for any  $m$  and  $s \leq 9$  the cases in which  $X \subset \mathbb{P}^2$  is not regular are:

- i)  $s = 2$ , and  $m \leq d \leq 2m - 2$ ;
- ii)  $s = 3$ , and  $\frac{3m}{2} \leq d \leq 2m - 2$ ;
- iii)  $s = 5$ , and  $2m \leq d \leq \frac{5m-2}{2}$ ;
- iv)  $s = 6$ , and  $\frac{12m}{5} \leq d \leq \frac{5m-2}{2}$ ;
- v)  $s = 7$ , and  $\frac{21m}{8} \leq d \leq \frac{8m-2}{3}$ ;
- vi)  $s = 8$ , and  $\frac{48m}{17} \leq d \leq \frac{17m-2}{6}$ .

Now we recall the notions of higher secant variety and  $k^{th}$  osculating variety.

**1.3. Definition.** Let  $V \subset \mathbb{P}^N$  be a closed irreducible projective variety; the  $(s-1)^{th}$  higher secant variety of  $V$  is the closure of the union of all linear spaces spanned by  $s$  points of  $V$ , and it will be denoted by  $V^s$ . Let  $\dim V = n$ ; the *expected dimension* for  $V^s$  is

$$(\dagger) \quad \expdim V^s = \min \{sn + s - 1, N\}$$

where the number  $sn + s - 1$  corresponds to  $\infty^{sn}$  choices of  $s$  points on  $V$ , plus  $\infty^{s-1}$  choices of a point on the  $\mathbb{P}^{s-1}$  spanned by the  $s$  points. When  $\dim V^s < \min\{sn + s - 1, N\}$ , the variety  $V^s$  is said to be *defective*, with *defect*  $\delta = \min\{sn + s - 1, N\} - \dim V^s$ .

**1.4. Definition.** Let  $V \subset \mathbb{P}^N$  be a variety, and let  $P \in V$  be a smooth point; we define the  $k^{th}$  osculating space to  $V$  at  $P$ , and we denote it by  $O_{k,V,P}$ , as the linear space defined by the vanishing of all linear forms  $L$  such that  $L|_V$  vanishes to order  $k+1$  on  $V$  at  $P$ . Let  $V_0 \subset V$  be the dense set of the smooth points where  $O_{k,V,P}$  has maximal dimension. The  $k^{th}$  osculating variety to  $V$  is defined as:

$$O_{k,V} = \overline{\bigcup_{P \in V_0} O_{k,V,P}}.$$

**1.5. Notation.** Set  $R = K[x, y, z] = \bigoplus R_d$ . Let  $V_d \subset \mathbb{P}^N$ ,  $N = \binom{d+2}{2} - 1$ , denote the  $d$ -ple Veronese embedding of  $\mathbb{P}^2$ , defined by the linear system  $R_d$  of all forms of a given degree  $d$ . Set  $O_{k,d} = O_{k,V_d}$ , so that the  $(s-1)^{th}$  higher secant variety to the  $k^{th}$  osculating variety to the Veronese surface  $V_d$  will be denoted by  $O_{k,d}^s$ .

**1.6. Remark.** We have (see [5], Lemma 2.3) that the dimension of  $O_{k,d}$  is always the expected one, that is

$$\dim O_{k,d} = \min \left\{ \binom{k+2}{2} + 1, \binom{d+2}{2} - 1 \right\}.$$

For  $d \leq k$  we immediately get  $O_{k,d} = \mathbb{P}^N$ , hence for  $d \leq k$  and for all  $s$ , we have  $O_{k,d}^s = \mathbb{P}^N$ .

Now we briefly recall how to associate to  $O_{k,d}^s$ , a zero dimensional scheme  $Y \subset \mathbb{P}^2$  (see [5], Remark 2.2).

**1.7. Remark.** Let  $\mathbb{P}^N = \mathbb{P}(R_d)$ , and let  $d \geq k+1$ . A form  $M \in R_d$  will denote, depending on the situation, a vector in  $R_d$  or a point in  $\mathbb{P}^N$ . We can view  $V_d$  as the image's closure of the map  $(\mathbb{P}^2)^* \rightarrow \mathbb{P}^N$ , where  $L \mapsto L^d$ ,  $L \in R_1$ . Hence

$$V_d = \{L^d, \quad L \in R_1\}.$$

At the point  $Q = L^d$  we have  $O_{k,V_d,Q} = \{L^{d-k}F, \quad F \in R_k\}$  and  $O_{k,d} = \bigcup_{Q \in V_d} O_{k,V_d,Q}$ . So we have:

$$O_{k,d} = \{L^{d-k}F, \quad L \in R_1, \quad F \in R_k\}$$

hence

$$O_{k,d}^s = \{L_1^{d-k}F_1 + \dots + L_s^{d-k}F_s, \quad L_i \in R_1, \quad F_i \in R_k, \quad i = 1, \dots, s\}.$$

Let  $P_i = L_i^{d-k}F_i$  be a generic point in  $O_{k,d}$ , and let  $T_{O_{k,d},P_i}$  be the tangent space of  $O_{k,d}$  at  $P_i$ . The affine cone over  $T_{O_{k,d},P_i}$  is  $W_i = \langle L_i^{d-k}R_k, L_i^{d-k-1}F_i R_1 \rangle$ .

Terracini's Lemma (see [13]) says that the tangent space of  $O_{k,d}^s$  at a generic point of  $\langle P_1, \dots, P_s \rangle$ , ( $P_1, \dots, P_s \in O_{k,d}$ ), is the span of the tangent spaces of  $O_{k,d}$  at  $P_i$  ( $1 \leq i \leq s$ ); if  $T_{O_{k,d},P_i} = \mathbb{P}(W_i)$ , then

$$\dim O_{k,d}^s = \dim \langle T_{O_{k,d},P_1}, \dots, T_{O_{k,d},P_s} \rangle = \dim \langle W_1, \dots, W_s \rangle - 1$$

Now consider the ortogonal space  $W_i^\perp \subset R_d$ , ( $1 \leq i \leq s$ ) via the apolarity action (for the definition of  $W_i^\perp$  see [5], Remark 2.5). It generates an ideal in  $R$  defining a scheme  $Z_i(k, d) \subset \mathbb{P}^2$ . Let  $Y$  be a generic union of  $s$  schemes  $Z_i(k, d)$  in  $\mathbb{P}^2$ , ( $1 \leq i \leq s$ ). Since

$$\dim \langle W_1, \dots, W_s \rangle - 1 = N - \dim[\langle W_1, \dots, W_s \rangle]^\perp = N - \dim(W_1^\perp \cap \dots \cap W_s^\perp) = N - h^0(\mathbb{P}^2, \mathcal{I}_Y(d)),$$

we have (see also [5], Remark 2.5):

$$(\ddagger) \quad \dim O_{k,d}^s = N - h^0(\mathbb{P}^2, \mathcal{I}_Y(d)) = H(Y, d) - 1$$

where  $H(Y, d)$  is the Hilbert function of  $Y$  in degree  $d$ .

Since for  $d = k + 1$ ,  $O_{k,d}^2 = \mathbb{P}^N$  (see [5], Proposition 3.4 C)), then for  $s \geq 2$  we immediately get:

**1.8. Proposition.** *For  $d = k + 1$  and  $s \geq 2$ , we have  $O_{k,d}^s = \mathbb{P}^N$ .*

For  $d \geq k + 2$ , the schemes  $Z_i(k, d)$  are zero-dimensional, and do not depend on  $d$ , in fact we have the following lemma (see [5], Lemmata 2.6, 2.7, 2.8):

**1.9. Lemma.** *Let  $Z(k, d) = Z_i(k, d)$  be one such scheme with support at  $P$ . For  $d \geq k + 2$ , we have:*

- i)  $(k + 1)P \subset Z(k, d) \subset (k + 2)P$ ;
- ii) the length of  $Z(k, d)$  is  $l(Z) = \binom{k+2}{2} + 2$ ;
- iii)  $Z(k, d) = Z(k, k + 2)$ .

Henceforth for  $d \geq k + 2$  we will denote  $Z(k, d)$  by  $Z(k)$ , or  $Z$ , if  $k$  is obvious by the context.

From  $(\ddagger)$  and the lemma above it follows that for  $d \geq k + 2$  in order to study the dimension of  $O_{k,d}^s$ , we only need to study the postulation of unions of generic schemes  $Z(k)$ .

**1.10. Remark.** Let  $d \geq k + 2$ . Recall that  $Z(k)$  is defined by the ideal generated by  $W^\perp \subset R_d$ , where  $W = \langle L^{d-k} R_k, L^{d-k-1} F R_1 \rangle$ , with  $L \in R_1$  and  $F \in R_k$ . Now we want to give a specialization of the scheme  $Z(k)$ : put  $L = x$  and  $F = y^k$ ; we get

$$W = \langle x^{d-k} R_k, x^{d-k-1} y^k R_1 \rangle$$

hence

$$W^\perp = \langle x^{d-k-1} y^{k-1} z^2, \dots, x^{d-k-1} y z^k, x^{d-k-1} z^{k+1}, x^{d-k-2} y^{k+2}, x^{d-k-2} y^{k+1} z, \dots, \\ x^{d-k-2} y z^{k+1}, x^{d-k-2} z^{k+2}, x^{d-k-3} y^{k+1}, x^{d-k-3} y^k z, \dots, x^{d-k-3} y z^k, x^{d-k-3} z^{k+1}, \dots, \\ x y^{d-1}, x y^{d-2} z, \dots, x y z^{d-2}, x z^{d-1}, y^d, y^{d-1} z, \dots, y z^{d-1}, z^d \rangle.$$

Let  $I$  be the ideal generated by  $W^\perp$ . By a direct computation, it is easy to show that the saturation of  $I$  is the ideal

$$(I)^{sat} = (y, z)^{k+1} \cap ((y, z)^{k+2} + (z^2))$$

that defines a scheme supported at a point of  $\mathbb{P}^2$ , whose structure is given by the union of its  $k^{th}$  infinitesimal neighbourhood, with the intersection of its  $(k + 1)^{th}$  infinitesimal neighbourhood with a double line.

**1.11. Notation.** We fix the following notation:

- i) let  $P_1, \dots, P_s$  be  $s$  generic points in  $\mathbb{P}^2$ ;
- ii) let  $X$  be the union of  $s$  generic  $(k + 1)$ -fat points in  $\mathbb{P}^2$ , with support in  $P_1, \dots, P_s$ ;

- iii) let  $T$  be the union of  $s$  generic  $(k+2)$ -fat points in  $\mathbb{P}^2$ , with support in  $P_1, \dots, P_s$ ;
- iv) let  $Z_i$  be a 0-dimensional scheme in  $\mathbb{P}^2$ , as defined in Remark 1.7, with support in  $P_i$ ;
- v) let  $Y = Z_1 + \dots + Z_s$ ;
- vi) denote by  $(k+1, k+2)P$  a 0-dimensional scheme whose defining ideal is  $\wp^{k+1} \cap (\wp^{k+2} + l^2)$  where  $\wp$  is the homogeneous ideal in  $R = K[x, y, z]$  of a point  $P \in \mathbb{P}^2$ , and  $l$  is the ideal of a generic line through  $P$ ; we call  $(k+1, k+2)P$  a  $(k+1, k+2)$  point;
- vii) let  $Z_i$  be a  $(k+1, k+2)$  point with support in  $P_i$ . By Remark 1.10, the scheme  $Z_i$  is a specialization of  $Z_i$ ;
- viii) let  $\mathcal{Y} = Z_1 + \dots + Z_s$  (so  $\mathcal{Y}$  is a specialization of the scheme  $Y$ ). We have

$$\deg \mathcal{Y} = \deg Y = s \left( \binom{k+2}{2} + 2 \right) = \deg X + 2s;$$

ix) if  $\mathcal{C} \subset \mathbb{P}^2$  is a curve, and  $Z$  is a zero-dimensional scheme, the scheme  $Z'$  defined by the ideal  $(I_Z : I_{\mathcal{C}})$  is called the residual of  $Z$  with respect to  $\mathcal{C}$ , and denoted by  $\text{Res}_{\mathcal{C}} Z$ .

In the following lemma we determine the subscheme of a  $(k+1, k+2)$  point with support in  $P$ , residual to a curve  $\mathcal{C}$ .

**1.12. Lemma.** *Let  $\mathcal{Z}$  be a  $(k+1, k+2)$  point, with support in  $P$  with defining ideal  $\wp^{k+1} \cap (\wp^{k+2} + l^2)$ , where  $\wp$  is the ideal of  $P$ , and  $l = (L)$  is the ideal of a generic line through  $P$ . Let  $\mathcal{C} \subset \mathbb{P}^2$  be a curve having at  $P$  a singularity of multiplicity  $m$ , and having  $L$  as tangent direction with multiplicity  $t$ . Then  $\text{Res}_{\mathcal{C}}(\mathcal{Z})$  is defined by the ideal*

$$I_{\text{Res}_{\mathcal{C}}(\mathcal{Z})} = \wp^{\max\{k+1-m; 0\}} \cap (\wp^{\max\{k+2-m; 0\}} + l^{\max\{2-t; 0\}}).$$

$\text{Res}_{\mathcal{C}}(\mathcal{Z})$  is a fat point, or a  $(k+1-m, k+2-m)$  point, except for  $m < k+1$  and  $t = 1$ , more precisely:

$$\text{Res}_{\mathcal{C}}(\mathcal{Z}) = \begin{cases} 0P & \text{for } m \geq k+2, \text{ or } m = k+1 \text{ and } t \geq 2 \\ 1P & \text{for } m = k+1 \text{ and } t \leq 1 \\ (k+1-m)P & \text{for } m < k+1 \text{ and } t \geq 2 \\ 2P & \text{for } m = k \text{ and } t = 0 \\ (k+1-m, k+2-m)P & \text{for } m < k \text{ and } t = 0 \end{cases}.$$

**Proof.** Without loss of generality, we assume that  $\wp = (x, y)$ ,  $L = x$ , and, by abuse of notation, that  $x, y$  are affine coordinates.

Let  $x^t f_1 + f_2 = 0$  be an equation defining the curve  $\mathcal{C}$ , where  $f_1$  is a homogeneous polynomial of degree  $m-t$ ,  $f_1 \notin (x)$ , and  $f_2 \in (x, y)^{m+1}$ . We have to prove that

$$((x, y)^{k+1} \cap ((x, y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x, y)^{\max\{k+1-m; 0\}} \cap ((x, y)^{\max\{k+2-m; 0\}} + (x^{\max\{2-t; 0\}})).$$

This is obvious for  $m \geq k+2$ , and for  $m = k+1, t \geq 2$ , since in these cases  $\text{Res}_{\mathcal{C}}(\mathcal{Z})$  is the emptyset.

Let  $m = k+1, t \leq 1$ . The equality above becomes

$$((x, y)^{k+1} \cap ((x, y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x, y).$$

“ $\subseteq$ ” : To prove this inclusion, let  $g = a + h$ ,  $a \in K$ ,  $h \in (x, y)$ . If  $g(x^t f_1 + f_2) = (a + h)(x^t f_1 + f_2) \in ((x, y)^{k+2} + (x^2))$ , since  $f_2 \in (x, y)^{m+1}$ ,  $hx^t f_1 \in (x, y)^{m+1}$ , and  $m+1 = k+2$ , it follows that

$ax^t f_1 \in ((x, y)^{k+2} + (x^2))$ . But  $f_1$  is a homogeneous polynomial of degree  $m - t$ ,  $f_1 \notin (x)$ ,  $t \leq 1$ , so it easily follows that  $a = 0$ , hence  $g \in (x, y)$ . The reverse inclusion is obvious.

Since  $I_{\text{Res}_C(\mathcal{Z})} = (x, y)$ , we have  $\text{Res}_C(\mathcal{Z}) = 1P$ .

Now, let  $m < k + 1$ ,  $t \geq 2$ . In this case we have to prove that

$$((x, y)^{k+1} \cap ((x, y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x, y)^{k+1-m}.$$

If  $g(x^t f_1 + f_2) \in (x, y)^{k+1}$ , it immediately follows that  $g \in (x, y)^{k+1-m}$ , and the reverse inclusion is obvious. Moreover, since  $I_{\text{Res}_C(\mathcal{Z})} = (x, y)^{k+1-m}$ , we have that  $\text{Res}_C(\mathcal{Z}) = (k + 1 - m)P$ .

Let  $m \leq k$ ,  $t \leq 1$ . Now we have to prove that

$$((x, y)^{k+1} \cap ((x, y)^{k+2} + (x^2))) : (x^t f_1 + f_2) = (x, y)^{k+1-m} \cap ((x, y)^{k+2-m} + (x^{2-t})).$$

“ $\subseteq$ ” : As in the previous case, if  $g(x^t f_1 + f_2) \in (x, y)^{k+1}$ , it follows that  $g \in (x, y)^{k+1-m}$ , so we can write

$$g = xg_1 + ay^{k+1-m} + g_2,$$

where  $g_1 \in (x, y)^{k-m}$  is homogeneous of degree  $k - m$ ,  $g_2 \in (x, y)^{k+2-m}$ ,  $a \in K$ . In order to prove that

$$g(x^t f_1 + f_2) = (xg_1 + ay^{k+1-m} + g_2)(x^t f_1 + f_2) \in ((x, y)^{k+2} + (x^2))$$

since  $g_2 x^t f_1$ , and  $f_2 \in (x, y)^{k+2}$ , it suffices to prove that

$$x^{t+1} g_1 f_1 + ax^t y^{k+1-m} f_1 \in ((x, y)^{k+2} + (x^2)).$$

Since  $x^{t+1} g_1 f_1 + ax^t y^{k+1-m} f_1$  is homogeneous of degree  $k+1$ , and  $f_1 \notin (x)$ , we get that  $x^{t+1} g_1 + ax^t y^{k+1-m} \in (x^2)$ . For  $t = 1$ , this implies  $a = 0$ , so  $g \in ((x, y)^{k+2-m} + (x))$ . For  $t = 0$  this implies  $a = 0$ , and  $g_1 \in (x)$ , so  $g \in ((x, y)^{k+2-m} + (x^2))$ .

“ $\supseteq$ ” : This inclusion is obvious.

So we have proved that, for  $m \leq k$  and  $t \leq 1$ :

$$I_{\text{Res}_C(\mathcal{Z})} = \begin{cases} (x, y)^{k+1-m} \cap ((x, y)^{k+2-m} + (x)) = (x, y)^{k+1-m} \cap (x, y)^{k+2-m} & \text{for } m \leq k \text{ and } t = 1 \\ (x, y) \cap ((x, y)^2 + (x^2)) = (x, y)^2 & \text{for } m = k \text{ and } t = 0, \\ (x, y)^{k+1-m} \cap ((x, y)^{k+2-m} + (x^2)) & \text{for } m < k \text{ and } t = 0 \end{cases},$$

hence for  $m = k$  and  $t = 0$  we have  $\text{Res}_C(\mathcal{Z}) = 2P$ , for  $m < k$  and  $t = 0$  we have  $\text{Res}_C(\mathcal{Z}) = (k + 1 - m, k + 2 - m)P$ , while for  $m \leq k$  and  $t = 1$ ,  $\text{Res}_C(\mathcal{Z})$  is the union of the fat point  $(k + 1 - m)P$  with the intersection of the line  $\{x = 0\}$  with the fat point  $(k + 2 - m)P$ .  $\square$

## 2. Osculating varieties to Veronese surface and some of their higher secant varieties.

In this section we will compute the dimension of  $O_{k,d}^s$  for  $3 \leq s \leq 6$  and  $s = 9$ .

**2.1. Remark.** We recall that for  $d \leq k + 1$  and  $s \geq 2$  (see Remark 1.6 and Proposition 1.8):

$$\dim O_{k,d}^s = N.$$

So we have to study the dimension of  $O_{k,d}^s$  only for  $d \geq k + 2$ . Since, for  $d \geq k + 2$  (see(‡))

$$\dim O_{k,d}^s = H(Y, d) - 1,$$

then, if we know the postulation of  $Y$ , we are done.

We wish to notice that, by (†), the expected dimension for  $O_{k,d}^s$  is

$$\expdim O_{k,d}^s = \min\{sn + s - 1, N\},$$

where  $n = \dim O_{k,d} = \min\left\{\binom{k+2}{2} + 1, \binom{d+2}{2} - 1\right\} = \min\left\{\binom{k+2}{2} + 1, N\right\} = \min\left\{\frac{\deg Y}{s} - 1, N\right\}$  (see Remark 1.6 and Lemma 1.9 ii)). Hence it easily follows that

$$\expdim O_{k,d}^s = \min\{\deg Y, N + 1\} - 1 = \exp H(Y, d) - 1$$

where  $\exp H(Y, d)$  is the expected value for  $H(Y, d)$ .

In the next lemmata we show that the postulation of  $Y$  is strictly related with the postulation of the specialized scheme  $\mathcal{Y}$ , and of the scheme of fat points  $X$ .

**2.2. Lemma.** *If the Hilbert function of the specialized scheme  $\mathcal{Y}$  in degree  $d$  is*

$$H(\mathcal{Y}, d) = \min\{H(X, d) + 2s, N + 1\},$$

*then*

$$H(Y, d) = \min\{H(X, d) + 2s, N + 1\}.$$

**Proof.** It follows from the obvious inequalities:  $H(\mathcal{Y}, d) \leq H(Y, d) \leq \min\{H(X, d) + 2s, N + 1\}$ .  $\square$

**2.3. Lemma.** *Let  $s > 2$ . Then:*

- i) for  $k = 1$ ,  $\mathcal{Y} = Y = (2, 3)P_1 + \dots + (2, 3)P_s$ , and  $H(\mathcal{Y}, d) = \min\{\deg Y, N + 1\}$ ;*
- ii) for  $k = 2$ ,  $\mathcal{Y} = (3, 4)P_1 + \dots + (3, 4)P_s$ , and  $H(\mathcal{Y}, d) = \min\{\deg Y, N + 1\}$ .*

**Proof.** *i)* If  $d = 2$  see [7], Proposition 3.3; for  $d = 3$  see [7], Proposition 4.5; for  $d \geq 4$  see [2], Theorem 1.

*ii)* follows from [3] Theorems 1 and 2.  $\square$

**2.4. Lemma.** *i) If  $H(\mathcal{Y}, d_0) = H(X, d_0) + 2s$ , then for every  $d \geq d_0$  we have*

$$H(\mathcal{Y}, d) = H(X, d) + 2s;$$

*ii) if  $(I_{\mathcal{Y}})_{d_0} = (0)$ , then for every  $d \leq d_0$  we have  $(I_{\mathcal{Y}})_d = (0)$ .*

**Proof.** *i)* Since  $X \subset \mathcal{Y}$  and  $H(\mathcal{Y}, d_0) = H(X, d_0) + 2s$ , then it easily follows that  $\dim(I_X/I_{\mathcal{Y}})_{d_0} = 2s$ . Therefore there are  $2s$  forms  $f_1, \dots, f_{2s} \in (I_X)_{d_0}$  linearly independent modulo  $(I_{\mathcal{Y}})_{d_0}$ . Let  $\{l = 0\}$  be a line not through any of the points  $P_1, \dots, P_s$ . The forms  $f_1 l^{d-d_0}, \dots, f_{2s} l^{d-d_0} \in (I_X)_d$  are linearly independent modulo  $(I_{\mathcal{Y}})_d$ , hence  $\dim(I_X/I_{\mathcal{Y}})_d \geq 2s$ , so we have  $H(\mathcal{Y}, d) \geq H(X, d) + 2s$ . Since obviously  $H(\mathcal{Y}, d) \leq H(X, d) + 2s$ , then the conclusion follows.



ii) Obvious.  $\square$

Now we will study the postulation of  $\mathcal{Y}$  for each  $s$  separately ( $s = 3, 4, 5, 6, 9$ ), but first we wish to mention the case  $s = 2$ .

**2.5. Proposition.** *For  $s = 2$  we have:*

$$H(\mathcal{Y}, d) = \begin{cases} \text{for } k = 1 : & \begin{cases} N + 1 & \text{if } d \leq 2 \\ H(T, d) = 9 < \exp H(\mathcal{Y}, d) & \text{if } d = 3 \\ H(X, d) + 4 = \deg Y & \text{if } d \geq 4 \end{cases} \\ \text{for } k = 2 : & \begin{cases} N + 1 & \text{if } d \leq 3 \\ H(T, d) = 14 < \exp H(\mathcal{Y}, d) & \text{if } d = 4 \\ H(X, d) + 4 = \deg Y & \text{if } d \geq 5 \end{cases} \\ \text{for } k \geq 3 : & \begin{cases} N + 1 & \text{if } d \leq k + 1 \\ H(T, d) = N < \exp H(\mathcal{Y}, d) & \text{if } d = k + 2 \\ H(X, d) + 4 < \exp H(\mathcal{Y}, d) & \text{if } k + 3 \leq d \leq 2k \\ H(X, d) + 4 = \deg Y & \text{if } d \geq 2k + 1 \end{cases} \end{cases}.$$

**Proof.** The case  $d \leq k + 1$  follows from Lemma 2.4 ii), and [5], Proposition 3.4, C).

For  $d = k + 2$ , observe that the line  $L$  through  $P_1$  and  $P_2$  is a component of multiplicity at least  $2(k + 1) - d = k$  for the curves defined by the forms both of  $(I_{\mathcal{Y}})_d$  and of  $(I_T)_d$ . Since  $\text{Res}_{kL}\mathcal{Y} = \text{Res}_{kL}T = 2P_1 + 2P_2$  (see Lemma 1.12), we get

$$\dim(I_{\mathcal{Y}})_{k+2} = \dim(I_T)_{k+2} = \dim(I_{2P_1+2P_2})_2 = 1$$

(the only curve is the  $(k + 2)$ -uple line through the two points). Thus  $H(\mathcal{Y}, d) = H(T, d)$ . Moreover, since  $T$  is not regular in degree  $k + 2$  (see Remark 1.2), we get  $H(\mathcal{Y}, d) < \exp H(\mathcal{Y}, d)$  (see [5], Corollary 3.5).

For  $k = 1, 2$  and  $d \geq k + 3$ , see [5], Corollary 3.8. For  $k \geq 3$ , and  $d \geq 2k + 1$  see [5], Proposition 3.9.

Now let  $k \geq 3$ , and  $k + 3 \leq d \leq 2k$ . For  $d = k + 3$  the line  $L$  through  $P_1$  and  $P_2$  is a component of multiplicity at least  $\nu = 2(k + 1) - d = k - 1$  for the curves defined by the forms of both  $(I_{\mathcal{Y}})_d$ , and  $(I_X)_d$ , hence from the case  $k = 1$ ,  $d = 4$ , we get

$$\dim(I_{\mathcal{Y}})_{k+3} = \dim(I_{\mathcal{Y}'})_{k+3-(k-1)} = \dim(I_{\mathcal{Y}'})_4 = 15 - 10 = 5, \quad \dim(I_X)_{k+3} = \dim(I_{X'})_4 = 9,$$

where  $\mathcal{Y}' = \text{Res}_{\nu L}\mathcal{Y} = (2, 3)P_1 + (2, 3)P_2$  (see Lemma 1.12), and  $X' = \text{Res}_{\nu L}X = 2P_1 + 2P_2$ .

It follows that  $H(\mathcal{Y}, k + 3) = H(X, k + 3) + 4$ . Hence by Lemma 2.4 i), for every  $d \geq k + 3$  we have

$$H(\mathcal{Y}, d) = H(X, d) + 4.$$

Since two  $(k + 1)$ -fat points impose independent conditions to curves of degree  $d$  if and only if  $d \geq 2k + 1$  (see Remark 1.2), then, for  $k + 3 \leq d \leq 2k$ , we have  $H(X, d) < \deg X$ , thus

$$H(\mathcal{Y}, d) = H(X, d) + 4 < \deg X + 4 = \deg Y.$$

Moreover, since for  $d = k + 3$ ,  $\dim(I_{\mathcal{Y}})_{k+3} = 5$ , then for  $d \geq k + 3$ ,  $\dim(I_{\mathcal{Y}})_d$  is positive, that is  $H(\mathcal{Y}, d) < \binom{d+2}{2}$ . It follows that if  $k + 3 \leq d \leq 2k$ , then  $H(\mathcal{Y}, d) < \min \left\{ \deg \mathcal{Y}, \binom{d+2}{2} \right\} = \exp H(\mathcal{Y}, d)$ .

(For  $k \geq 3$ , and  $k + 3 \leq d \leq 2k$ , see also [5], Proposition 3.10).  $\square$

**2.6. Proposition.** *For  $s = 3$  we have:*

$$i) \quad H(\mathcal{Y}, d) = \begin{cases} N + 1 & \text{if } d \leq \lceil \frac{3(k+1)}{2} \rceil \\ H(X, d) + 6 < \deg Y & \text{if } \lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k \\ H(X, d) + 6 = \deg Y & \text{if } d \geq \max\{\lceil \frac{3(k+1)}{2} \rceil + 1; 2k + 1\} \end{cases}.$$

$$ii) \quad H(\mathcal{Y}, d) < \exp H(\mathcal{Y}, d) \quad \text{iff} \quad \begin{cases} \lceil \frac{3(k+1)}{2} \rceil + 2 \leq d \leq 2k & \text{if } k + 1 \text{ is even} \\ \lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k & \text{if } k + 1 \text{ is odd} \end{cases}.$$

**Proof.** *i)* In case  $d \leq \lceil \frac{3(k+1)}{2} \rceil$ , it suffices to prove that  $(I_{\mathcal{Y}})_d = (0)$  for  $d = \lceil \frac{3(k+1)}{2} \rceil$ .

Let  $\mathcal{C}$  be the curve formed by the three lines  $P_1P_2$ ,  $P_1P_3$ ,  $P_2P_3$ . For  $d = \lceil \frac{3(k+1)}{2} \rceil$ , the curve  $\mathcal{C}$  is a fixed component, of multiplicity at least

$$\nu = 2(k+1) - d = \begin{cases} \frac{k+1}{2} & \text{if } k+1 \text{ is even} \\ \frac{k}{2} & \text{if } k+1 \text{ is odd} \end{cases}$$

for the curves defined by the forms of  $(I_{\mathcal{Y}})_d$ , so we have (see Lemma 1.12)

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d-3\nu}$$

where

$$\mathcal{Y}' = \text{Res}_{\nu\mathcal{C}}\mathcal{Y} = \begin{cases} P_1 + P_2 + P_3 & \text{if } k+1 \text{ is even} \\ 2P_1 + 2P_2 + 2P_3 & \text{if } k+1 \text{ is odd} \end{cases}, \quad d - 3\nu = \begin{cases} 0 & \text{if } k+1 \text{ is even} \\ 2 & \text{if } k+1 \text{ is odd} \end{cases}.$$

It immediately follows that  $(I_{\mathcal{Y}})_d = (0)$ .

Now let  $d \geq \lceil \frac{3(k+1)}{2} \rceil + 1$ . In order to prove that  $H(\mathcal{Y}, d) = H(X, d) + 6$ , by Lemma 2.4 it suffices to prove that  $H(\mathcal{Y}, d) = H(X, d) + 6$  for  $d = \lceil \frac{3(k+1)}{2} \rceil + 1$ .

Let  $d = \lceil \frac{3(k+1)}{2} \rceil + 1$ . The curve  $\mathcal{C}$  is a fixed component, with multiplicity at least

$$\nu = 2(k+1) - d = \begin{cases} \frac{k-1}{2} & \text{if } k+1 \text{ is even} \\ \frac{k-2}{2} & \text{if } k+1 \text{ is odd} \end{cases}$$

for the curves defined by the forms of both  $(I_{\mathcal{Y}})_d$  and  $(I_X)_d$ , then we have

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d-3\nu}, \quad \dim(I_X)_d = \dim(I_{X'})_{d-3\nu}$$

where (see Lemma 1.12)

$$\begin{aligned} d - 3\nu &= \begin{cases} 4 & \text{if } k+1 \text{ is even} \\ 6 & \text{if } k+1 \text{ is odd} \end{cases}, \\ \mathcal{Y}' &= \text{Res}_{\nu\mathcal{C}}\mathcal{Y} = \begin{cases} (2, 3)P_1 + (2, 3)P_2 + (2, 3)P_3 & \text{if } k+1 \text{ is even} \\ (3, 4)P_1 + (3, 4)P_2 + (3, 4)P_3 & \text{if } k+1 \text{ is odd} \end{cases}, \\ X' &= \begin{cases} 2P_1 + 2P_2 + 2P_3 & \text{if } k+1 \text{ is even} \\ 3P_1 + 3P_2 + 3P_3 & \text{if } k+1 \text{ is odd} \end{cases}. \end{aligned}$$

Since it is well known that  $\dim(I_{2P_1+2P_2+2P_3})_4 = 6$  and  $\dim(I_{3P_1+3P_2+3P_3})_6 = 10$ , we have

$$\dim(I_{X'})_{d-3\nu} = \begin{cases} 6 & \text{if } k+1 \text{ is even} \\ 10 & \text{if } k+1 \text{ is odd} \end{cases},$$

moreover, by Lemma 2.3 we get that

$$\dim(I_{Y'})_{d-3\nu} = \begin{cases} 0 & \text{if } k+1 \text{ is even} \\ 4 & \text{if } k+1 \text{ is odd} \end{cases}.$$

It follows that  $\dim(I_X)_d - \dim(I_Y)_d = 6$ , hence  $H(\mathcal{Y}, d) - H(X, d) = 6$ .

Since three  $(k+1)$ -fat points impose independent conditions to curves of degree  $d$  if and only if  $d \geq 2k+1$  (see Remark 1.2), then for  $\lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k$  we have  $H(X, d) < \deg X$ , while if  $d \geq \max \left\{ \lceil \frac{3(k+1)}{2} \rceil + 1; 2k+1 \right\}$ , then  $H(X, d) = \deg X$ . Since  $\deg Y = \deg X + 6$  we get:

$$H(\mathcal{Y}, d) = \begin{cases} H(X, d) + 6 < \deg Y & \text{if } \lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k \\ H(X, d) + 6 = \deg Y & \text{if } d \geq \max \left\{ \lceil \frac{3(k+1)}{2} \rceil + 1; 2k+1 \right\} \end{cases}.$$

ii) For  $d \leq \lceil \frac{3(k+1)}{2} \rceil$ , or  $d \geq \max \left\{ \lceil \frac{3(k+1)}{2} \rceil + 1; 2k+1 \right\}$ , from i) we have  $H(\mathcal{Y}, d) = \exp H(\mathcal{Y}, d)$ .

If  $k+1$  is even, and  $d = \lceil \frac{3(k+1)}{2} \rceil + 1$ , then  $\dim(I_Y)_d = 0$ , hence  $H(\mathcal{Y}, d) = \binom{d+2}{2}$ , the expected one.

If  $k+1$  is even, and  $d = \lceil \frac{3(k+1)}{2} \rceil + 2$ , from i), since  $\dim(I_X)_{d-1} = 6$  implies  $\dim(I_X)_d > 6$ , we have:

$$\dim(I_Y)_d = \binom{d+2}{2} - H(\mathcal{Y}, d) = \binom{d+2}{2} - H(X, d) - 6 = \dim(I_X)_d - 6 > 0.$$

Hence, if  $k+1$  is even, for  $d = \lceil \frac{3(k+1)}{2} \rceil + 2$ , and so also for  $d \geq \lceil \frac{3(k+1)}{2} \rceil + 2$ , we have  $\dim(I_Y)_d > 0$ , that is  $H(\mathcal{Y}, d) < \binom{d+2}{2}$ . Since, by i), if  $\lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k$ , then  $H(\mathcal{Y}, d) < \deg Y$ , it follows that for  $\lceil \frac{3(k+1)}{2} \rceil + 2 \leq d \leq 2k$  we have  $H(\mathcal{Y}, d) < \min \left\{ \deg Y, \binom{d+2}{2} \right\} = \exp H(\mathcal{Y}, d)$ .

If  $k+1$  is odd and  $d \geq \lceil \frac{3(k+1)}{2} \rceil + 1$ , from the proof of i) we get  $\dim(I_Y)_d > 0$ , that is  $H(\mathcal{Y}, d) < \binom{d+2}{2}$ . Moreover, by i), if  $\lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k$ , then  $H(\mathcal{Y}, d) < \deg Y$ , and the conclusion immediately follows.  $\square$

**2.7. Proposition.** *For  $s = 4$  we have:*

$$H(\mathcal{Y}, d) = \begin{cases} \text{for } k \leq 6 : & \begin{cases} N+1 & \text{if } d \leq 2k+2 \\ H(X, d) + 8 = \deg Y & \text{if } d \geq 2k+3 \end{cases} \\ \text{for } k \geq 6 : & \begin{cases} N+1 & \text{if } d \leq 2k+1 \\ H(X, d) + 8 = \deg Y & \text{if } d \geq 2k+2 \end{cases} \end{cases}.$$

**Proof.** If  $d \leq 2k+1$ , by Bezout Theorem, each element of  $(I_Y)_d$  is divisible by every form defining an irreducible conic through  $P_1, \dots, P_4$ , hence  $(I_Y)_d = (0)$ .

Let  $d = 2k+2$ . Recall that the ideal of the scheme  $\mathcal{Z}_i$  is  $\wp_i^{k+1} \cap (\wp_i^{k+2} + l_i^2)$ , where  $l_i$  defines a generic line  $L_i$  through  $P_i$  ( $1 \leq i \leq 4$ ) such that  $\deg(\mathcal{Y} \cap L_i) = k+2$ . Let  $\mathcal{C}_i$  be the conic through  $P_1, \dots, P_4$ , tangent in  $P_i$  to  $L_i$ . For the genericity of the  $L_i$ 's, the conics  $\mathcal{C}_1, \dots, \mathcal{C}_4$  are irreducible and distinct. Bezout's Theorem implies that each conic  $\mathcal{C}_i$  is a component of any curve defined by the forms of  $(I_Y)_d$ . By Lemma 1.12 we can determine  $I_{\text{Res}_{\mathcal{C}_1+\dots+\mathcal{C}_4}\mathcal{Y}}$ , and it is an easy computation (which will be omitted) that the intersection multiplicities of the curves defined by the forms of  $(I_{\text{Res}_{\mathcal{C}_1+\dots+\mathcal{C}_4}\mathcal{Y}})_{d-8}$  with a conic  $\mathcal{C}_i$ , is bigger than  $2(d-8)$ . Hence by Bezout's Theorem we get that each conic  $\mathcal{C}_i$  is a component with multiplicity at least 2 of any curve

defined by the forms of  $(I_{\mathcal{Y}})_d$ . So these curves have a component of degree 16. It follows that, if  $(I_{\mathcal{Y}})_d \neq (0)$ , then  $d \geq 16$ , that is  $k \geq 7$ . Thus, for  $k \leq 6$ , we have  $(I_{\mathcal{Y}})_d = (0)$ , that is  $H(\mathcal{Y}, d) = N + 1$ . Observe that for  $k = 6$ , we have  $N + 1 = H(X, d) + 8 = \deg Y$ , in fact in this case  $d = 2k + 2 = 14$ ,  $N + 1 = \binom{16}{2} = 120$ , and, since four 7-fat points impose independent conditions to curves of degree 14 (see Remark 1.2), then  $H(X, d) = 112$ . If  $k \geq 7$  we have

$$\dim(I_{\mathcal{Y}})_{2k+2} = \dim(I_{\mathcal{Y}'})_{2k+2-16},$$

where  $\mathcal{Y}' = \text{Res}_{\mathcal{C}_1 + \dots + 2\mathcal{C}_4} \mathcal{Y} = (k-7)P_1 + \dots + (k-7)P_4$  is a scheme of four  $(k-7)$ -fat points (see Lemma 1.12). Since  $\mathcal{Y}'$  imposes independent conditions to curves of degree  $2k-14$  (see Remark 1.2), then  $H(\mathcal{Y}, 2k+2) = \binom{2k+4}{2} - \dim(I_{\mathcal{Y}})_{2k+2} = \binom{2k+4}{2} - \dim(I_{\mathcal{Y}'})_{2k-14} = \binom{2k+4}{2} - \binom{2k-12}{2} + 4\binom{k-6}{2} = 4\binom{k+2}{2} + 8 = H(X, 2k+2) + 8 = \deg Y$ .

Now let  $d \geq 2k+3$ . It suffices to prove that  $H(\mathcal{Y}, 2k+3) = H(X, 2k+3) + 8 = \deg Y$  (see Lemma 2.4 i)), so let  $d = 2k+3$ . By induction on  $k$ . For  $k = 1$  see Lemma 2.3. Let  $k \geq 2$ . Let  $\mathcal{C}$  be an irreducible conic through  $P_1, \dots, P_4$ , and let  $Q_1, Q_2, Q_3$  be three points on  $\mathcal{C}$ . Let  $\tilde{\mathcal{Y}} = \mathcal{Y} + Q_1 + Q_2 + Q_3$ . By Bezout's Theorem, the conic  $\mathcal{C}$  is a fixed component for the curves of degree  $2k+3$  through  $\tilde{\mathcal{Y}}$ , then

$$\dim(I_{\tilde{\mathcal{Y}}})_{2k+3} = \dim(I_{\tilde{\mathcal{Y}'}})_{2k+1} = \binom{2k+3}{2} - H(\tilde{\mathcal{Y}'}, 2k+1),$$

where  $\tilde{\mathcal{Y}}' = \text{Res}_{\mathcal{C}} \tilde{\mathcal{Y}} = \text{Res}_{\mathcal{C}} \mathcal{Y} = \sum_{i=1}^4 (k, k+1)P_i$  (see Lemma 1.12). By the inductive hypothesis we have that  $H(\tilde{\mathcal{Y}}', 2k+1) = \deg \tilde{\mathcal{Y}}' = 4\binom{k+1}{2} + 8$ , hence

$$H(\tilde{\mathcal{Y}}, 2k+3) = \binom{2k+5}{2} - \binom{2k+3}{2} + 4\binom{k+1}{2} + 8 = \deg \mathcal{Y} + 3 = \deg \tilde{\mathcal{Y}}.$$

Hence  $\tilde{\mathcal{Y}}$  imposes independent conditions to curves of degree  $2k+3$ . Since  $\mathcal{Y} \subset \tilde{\mathcal{Y}}$ , then also  $\mathcal{Y}$  imposes independent conditions to curves of degree  $2k+3$ , that is  $H(\mathcal{Y}, 2k+3) = \deg \mathcal{Y} = \deg Y$ . □

**2.8. Proposition.** *For  $s = 5$  we have:*

$$H(\mathcal{Y}, d) = \begin{cases} N + 1 & \text{if } d \leq 2k + 3 \\ H(X, d) + 10 < \exp H(\mathcal{Y}, d) & \text{if } 2k + 4 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1 \\ H(X, d) + 10 = \deg Y & \text{if } d \geq \max \left\{ 2k + 4; \lfloor \frac{5(k+1)}{2} \rfloor \right\} \end{cases}.$$

**Proof.** Let  $d \leq 2k+3$ . If we prove that  $(I_{\mathcal{Y}})_d = (0)$  for  $d = 2k+3$  we are done. So let  $d = 2k+3$ . For  $k = 1$  see Lemma 2.3. Let  $k \geq 2$ . Any curve defined by a nonzero element of  $(I_X)_d$  has the conic  $\mathcal{C}$  through  $P_1, \dots, P_5$  as a component of multiplicity at least  $5(k+1) - 2d = k-1$ , where  $X$  is the fat point subscheme of 5 points of multiplicity  $k+1$ , hence the same is true for  $\mathcal{Y}$  in place of  $X$ , since  $X \subset \mathcal{Y}$ , so we have:

$$\dim(I_{\mathcal{Y}})_{2k+3} = \dim(I_{\mathcal{Y}'})_{2k+3-2(k-1)} = \dim(I_{\mathcal{Y}'})_5,$$

where, by Lemma 1.12,  $\mathcal{Y}' = \text{Res}_{(k-1)\mathcal{C}} \mathcal{Y} = (2, 3)P_1 + \dots + (2, 3)P_5$ . Since, by Lemma 2.3 i),  $\dim(I_{\mathcal{Y}'})_5 = 0$ , then the conclusion follows.

Now let  $d \geq 2k+4$ . We have to prove that

$$H(\mathcal{Y}, d) = H(X, d) + 10.$$

By Lemma 2.4, it suffices to prove that  $H(\mathcal{Y}, d) = H(X, d) + 10$  for  $d = 2k + 4$ , so let  $d = 2k + 4$ . For  $k = 1, 2$  see Lemma 2.3. If  $k = 3$  (hence  $d = 10$ ), let  $Q$  be a point on the conic  $\mathcal{C}$  through  $P_1, \dots, P_5$ . The scheme  $\mathcal{Y} + Q$  imposes independent conditions to curves of degree 10. In fact, since the conic  $\mathcal{C}$  is a fixed locus for  $(I_{\mathcal{Y}+Q})_{10}$ , from the case  $k = 2$  we get:

$$\dim(I_{\mathcal{Y}+Q})_{10} = \dim(I_{\mathcal{Y}'})_8 = \binom{8+2}{2} - 5(8) = 5 = \binom{10+2}{2} - 5(12) - 1 = \binom{10+2}{2} - \deg(\mathcal{Y} + Q),$$

where  $\mathcal{Y}' = \text{Res}_{\mathcal{C}}(\mathcal{Y} + Q) = (3, 4)P_1 + \dots + (3, 4)P_5$  (see Lemma 1.12). Since  $\mathcal{Y} + Q$  imposes independent conditions to curves of degree 10, then  $\mathcal{Y}$  and  $X$  also do the same. It follows that

$$H(\mathcal{Y}, 10) = \deg Y = \deg X + 10 = H(X, 10) + 10.$$

For  $k \geq 4$ , since  $\mathcal{C}$  is a fixed component with multiplicity at least  $(k - 3)$  for curves defined both by  $(I_{\mathcal{Y}})_{2k+4}$  and by  $(I_X)_{2k+4}$ , it follows that

$$\dim(I_{\mathcal{Y}})_{2k+4} = \dim(I_{\mathcal{Y}'} )_{2k+4-2(k-3)} = \dim(I_{\mathcal{Y}'} )_{10}, \quad \dim(I_X)_{2k+4} = \dim(I_{X'})_{10},$$

where (see Lemma 1.12)

$$\mathcal{Y}' = \text{Res}_{(k-3)\mathcal{C}}\mathcal{Y} = (4, 5)P_1 + \dots + (4, 5)P_5, \quad X' = \text{Res}_{(k-3)\mathcal{C}}4P_1 + \dots + 4P_5.$$

From the case  $k = 3$  it follows that

$$\dim(I_{\mathcal{Y}})_{2k+4} = 6, \quad \dim(I_X)_{2k+4} = 16,$$

hence  $H(\mathcal{Y}, d) = H(X, d) + 10$ .

So we have proved that for  $d \geq 2k + 4$

$$H(\mathcal{Y}, d) = H(X, d) + 10.$$

Now, since  $\dim(I_{\mathcal{Y}})_{2k+4}$  is positive, then  $H(\mathcal{Y}, d) < \binom{d+2}{2}$  for any  $d \geq 2k + 4$ . Moreover, since five generic  $(k + 1)$ -fat points impose independent conditions to curves of degree  $d$  if and only if  $d \geq \lfloor \frac{5(k+1)}{2} \rfloor$  (see Remark 1.2), then for  $2k + 4 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$ , we have  $H(X, d) < \deg X$ , hence

$$H(\mathcal{Y}, d) = H(X, d) + 10 < \min \left\{ \deg X + 10, \binom{d+2}{2} \right\} = \min \left\{ \deg Y, \binom{d+2}{2} \right\} = \exp H(\mathcal{Y}, d).$$

If  $d \geq \max \left\{ 2k + 4; \lfloor \frac{5(k+1)}{2} \rfloor \right\}$ , then  $H(X, d) = \deg X$ , so  $H(\mathcal{Y}, d) = \deg Y$ . □

**2.9. Proposition.** *For  $s = 6$  we have:*

$$H(\mathcal{Y}, d) = \begin{cases} \text{for } k = 1 : & \begin{cases} N + 1 & \text{if } d \leq 6 \\ H(X, d) + 12 = \deg Y & \text{if } d \geq 7 \end{cases} \\ \text{for } k = 2 : & \begin{cases} N + 1 & \text{if } d \leq 8 \\ H(X, d) + 12 = \deg Y & \text{if } d \geq 9 \end{cases} \\ \text{for } k \geq 3 \\ k \equiv 2 \pmod{5} : & \begin{cases} N + 1 & \text{if } d \leq \lceil \frac{12(k+1)}{5} \rceil - 1 \\ H(X, d) + 12 < \exp H(\mathcal{Y}, d) & \text{if } \lceil \frac{12(k+1)}{5} \rceil \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1 \\ H(X, d) + 12 = \deg Y & \text{if } d \geq \max \left\{ \lceil \frac{12(k+1)}{5} \rceil; \lfloor \frac{5(k+1)}{2} \rfloor \right\} \end{cases} \\ \text{for } k \geq 3 \\ k \not\equiv 2 \pmod{5} : & \begin{cases} N + 1 & \text{if } d \leq \lceil \frac{12(k+1)}{5} \rceil \\ H(X, d) + 12 < \exp H(\mathcal{Y}, d) & \text{if } \lceil \frac{12(k+1)}{5} \rceil + 1 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1 \\ H(X, d) + 12 = \deg Y & \text{if } d \geq \max \left\{ \lceil \frac{12(k+1)}{5} \rceil + 1; \lfloor \frac{5(k+1)}{2} \rfloor \right\} \end{cases} \end{cases}.$$

**Proof.** We start by proving four particular cases, that we need later in the proof.

**2.10. Lemma.** *We have:*

- i)  $\dim(I_{(8,9)P_1+\dots+(8,9)P_6})_{20} = 3$ ;
- ii)  $\dim(I_{(6,7)P_1+\dots+(6,7)P_6})_{15} = 0$ ;
- iii)  $\dim(I_{(5,6)P_1+\dots+(5,6)P_6})_{13} = 3$ ;
- iv)  $\dim(I_{(4,5)P_1+\dots+(4,5)P_6})_{11} = 6$ .

**Proof.** Though all of the above equalities can be checked using CoCoA (see [6]), we prove i) by specializing the scheme  $\mathcal{Y}$ . The proofs of ii), iii), and iv) by using a specialization of  $\mathcal{Y}$  are similar, and we left them to the reader.

i) Let  $Q \in \mathbb{P}^2$  be a generic point, and let  $\mathcal{F} = \{F = 0\}$  be a rational integral curve of degree 5 passing through  $Q$ , and having at each  $P_i$ ,  $1 \leq i \leq 6$ , an ordinary singularity of multiplicity 2, (so  $F \in (I_{2P_1+\dots+2P_6})_5$ ), and let  $\{\tilde{l}_i = 0\}$  be one of the two distinct lines contained in the tangent space  $T_{\mathcal{F},P_i}$  to  $\mathcal{F}$  at the point  $P_i$ . Recall that the defining ideal of  $\mathcal{Y} = (8,9)P_1 + \dots + (8,9)P_6$  is

$$I_{\mathcal{Y}} = (\wp_1^8 \cap (\wp_1^9 + l_1^2)) \cap \dots \cap (\wp_6^8 \cap (\wp_6^9 + l_6^2)).$$

Specialize the scheme  $\mathcal{Y}$  putting  $l_i = \tilde{l}_i$  for  $i = 1, 2, 3, 4$ , and let  $\mathcal{Y}^*$  be such specialization of  $\mathcal{Y}$ . Since the expected dimension of  $(I_{\mathcal{Y}})_{20}$  is  $\binom{20+2}{2} - \deg \mathcal{Y} = 231 - 228 = 3$ , then if we prove that  $\dim(I_{\mathcal{Y}^*})_{20} = 3$ , we are done. It is easy to see that the curves defined by the forms of  $(I_{\mathcal{Y}^*+Q})_{20}$  have the quintic  $\mathcal{F}$  as fixed component with multiplicity at least 2, hence

$$\dim(I_{\mathcal{Y}^*+Q})_{20} = \dim(I_{\mathcal{W}})_{10}$$

where  $\mathcal{W} = \text{Res}_{2\mathcal{F}}(\mathcal{Y}^* + Q) = 4P_1 + 4P_2 + 4P_3 + 4P_4 + (4,5)P_5 + (4,5)P_6$ . Now let  $\mathcal{W}^*$  be a specialization of  $\mathcal{W}$  obtained by putting  $l_i = \tilde{l}_i$  for  $i = 5, 6$ . Since the quintic  $\mathcal{F}$  is as fixed component with multiplicity at least 2 for  $(I_{\mathcal{W}^*+Q})_{10}$ , and since  $\text{Res}_{2\mathcal{F}}(\mathcal{W}^* + Q) = \emptyset$  (see Lemma 1.12) we have

$$\dim(I_{\mathcal{W}^*+Q})_{10} = \dim(I_{\text{Res}_{2\mathcal{F}}(\mathcal{W}^*+Q)})_0 = 1.$$

Thus for the specialized scheme  $\mathcal{W}^*$  we have  $\dim(I_{\mathcal{W}^*})_{10} = 2 = \binom{10+2}{2} - \deg \mathcal{W}^*$ . Then  $\mathcal{W}^*$ , and so  $\mathcal{W}$  also, imposes independent conditions to curves of degree 10. It follows that  $\dim(I_{\mathcal{W}})_{10} = 2$ . So  $\dim(I_{\mathcal{Y}^*+Q})_{20} = 2$ , hence  $\dim(I_{\mathcal{Y}^*})_{20} = 3$ , and we are done.  $\square$

Now let  $k+1 = 5q+r$ , ( $0 \leq r \leq 4$ ). Thus  $k \equiv 2 \pmod{5}$  iff  $r = 3$ .

For  $k = 1, 2$  see Lemma 2.3.

Let  $k \geq 3$ . Let  $\mathcal{C}_i$  be the conic through  $P_1, \dots, \widehat{P}_i, \dots, P_6$ , ( $i = 1, \dots, 6$ ), and let  $\mathcal{C} = \sum_{i=1}^6 \mathcal{C}_i$ . Observe that if  $2d < 5(k+1)$ , then the curves defined by the forms of  $(I_{\mathcal{Y}})_d$ , and by the forms of  $(I_X)_d$  have the six conics  $\mathcal{C}_i$  as fixed components with multiplicity at least  $\nu = 5(k+1) - 2d$ .

Then

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d-12\nu}, \quad \dim(I_X)_d = \dim(I_{X'})_{d-12\nu},$$

where

$$\mathcal{Y}' = \text{Res}_{\nu\mathcal{C}}\mathcal{Y} = (k+1-5\nu, k+2-5\nu)P_1 + \dots + (k+1-5\nu, k+2-5\nu)P_6,$$

$$X' = \text{Res}_{\nu\mathcal{C}} X = (k+1-5\nu)P_1 + \cdots + (k+1-5\nu)P_6.$$

We split the proof in four cases.

*Case 1):*  $k \equiv 2 \pmod{5}$ , and  $d \leq \lceil \frac{12(k+1)}{5} \rceil - 1 = 12q + 7$ .

In this case it suffices to prove that  $(I_{\mathcal{Y}})_d = (0)$  for  $d = 12q + 7$ . Since  $2d = 2(12q + 7) < 5(k+1) = 5(5q+3)$ , then the curves defined by the forms of  $(I_{\mathcal{Y}})_d$  should have a fixed locus of degree  $12\nu = 12q + 12$ , and this is impossible, since  $d = 12q + 7$ . It follows that  $(I_{\mathcal{Y}})_d = (0)$ .

*Case 2):*  $k \equiv 2 \pmod{5}$ , and  $d \geq \lceil \frac{12(k+1)}{5} \rceil = 12q + 8$ .

First we will prove that

$$H(\mathcal{Y}, d) = H(X, d) + 12.$$

By Lemma 2.4, it suffices to prove that  $H(\mathcal{Y}, d) = H(X, d) + 12$ , for  $d = 12q + 8$ . Since  $k \geq 3$ , and  $k+1 = 5q+3$ , then we have  $q \geq 1$ . Let  $q = 1$ , so  $d = 20$ ,  $k+1 = 8$ ,  $\mathcal{Y} = (8,9)P_1 + \cdots + (8,9)P_6$ , and  $X = 8P_1 + \cdots + 8P_6$ . Since  $\dim(I_{(8,9)P_1 + \cdots + (8,9)P_6})_{20} = 3$  (see Lemma 2.10 i)), and six 8-fat points impose independent conditions to curves of degree 20 (see Remark 1.2), we have  $\dim(I_X)_{20} = 15$ . It follows that  $H(\mathcal{Y}, d) = H(X, d) + 12$ . If  $q > 1$ , then  $\nu\mathcal{C} = \sum_{i=1}^6 \nu\mathcal{C}_i$  is a fixed locus for  $(I_{\mathcal{Y}})_d$  and  $(I_X)_d$ . Since  $\nu = 5(k+1) - 2d = 5(5q+3) - 2(12q+8) = q-1$ , we have  $d - 12\nu = 12q + 8 - 12(q-1) = 20$ , and  $k+1-5\nu = 5q+3-5(q-1) = 8$ . So

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{20} = 3, \quad \dim(I_X)_d = \dim(I_{X'})_{20} = 15,$$

where  $\mathcal{Y}' = \text{Res}_{\nu\mathcal{C}} \mathcal{Y} = (8,9)P_1 + \cdots + (8,9)P_6$ ,  $X' = \text{Res}_{\nu\mathcal{C}} X = 8P_1 + \cdots + 8P_6$ .

So we have proved that  $H(\mathcal{Y}, d) = H(X, d) + 12$ .

Now, since for  $d = 12q + 8$ ,  $\dim(I_{\mathcal{Y}})_d$  is positive (and in fact it is equal to  $\dim(I_{\mathcal{Y}'})_{20} = 3$ ), then  $H(\mathcal{Y}, d) < \binom{d+2}{2}$  for any  $d \geq 12q + 8$ .

Since six generic  $(k+1)$ -fat points impose independent conditions to curves of degree  $d$  if and only if  $d \geq \lfloor \frac{5(k+1)}{2} \rfloor$  (see Remark 1.2), then for  $12q + 8 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$ , we have  $H(X, d) < \deg X$ , hence

$$H(\mathcal{Y}, d) = H(X, d) + 12 < \min \left\{ \deg X + 12, \binom{d+2}{2} \right\} = \min \left\{ \deg Y, \binom{d+2}{2} \right\} = \exp H(\mathcal{Y}, d).$$

While for  $d \geq \max \left\{ 12q + 8; \lfloor \frac{5(k+1)}{2} \rfloor \right\}$ , we have  $H(X, d) = \deg X$ , so  $H(\mathcal{Y}, d) = H(X, d) + 12 = \deg X + 12 = \deg Y$ . That is enough to finish the proof of this case.

*Case 3):*  $k \not\equiv 2 \pmod{5}$ , and  $d \leq \lceil \frac{12(k+1)}{5} \rceil$ .

By Lemma 2.4 we have only to prove that  $H(\mathcal{Y}, d) = N + 1$  for  $d = \lceil \frac{12(k+1)}{5} \rceil = 12q + \lceil \frac{12r}{5} \rceil$ . Since  $k \geq 3$ , we have  $k+1 = 5q+r \geq 4$ , hence  $q \geq \frac{4-r}{5}$ . As above, let  $\nu = 5(k+1) - 2d$ ,  $\mathcal{Y}' = \text{Res}_{\nu\mathcal{C}} \mathcal{Y}$ , and let  $d' = d - 12\nu$ . We have:

$r$	$k + 1$	$d$	$\nu$	$\mathcal{Y}'$	$d'$
0	$5q$	$12q$	$q > 0$	$P_1 + \cdots + P_6$	0
1	$5q + 1$	$12q + 3$	$q - 1 \geq 0$	$(6,7)P_1 + \cdots + (6,7)P_6$	15
2	$5q + 2$	$12q + 5$	$q > 0$	$(2,3)P_1 + \cdots + (2,3)P_6$	5
4	$5q + 4$	$12q + 10$	$q \geq 0$	$(4,5)P_1 + \cdots + (4,5)P_6$	10

Since for  $\nu = 0$ , we have  $\mathcal{Y}' = \mathcal{Y}$  and  $d' = d$ , then for every  $\nu \geq 0$  we have:

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'}_{d'}).$$

Now we will prove that  $\dim(I_{\mathcal{Y}'}_{d'}) = 0$ .

For  $r = 0$  it is obvious. For  $r = 2$  see Lemma 2.3. For  $r = 1$  by Lemma 2.10 ii), we have  $\dim(I_{(6,7)P_1+\dots+(6,7)P_6})_{15} = 0$ . For  $r = 4$ , let  $\mathcal{F} = \{F = 0\}$  be a rational integral curve of degree 5 having at each  $P_i$  ( $1 \leq i \leq 6$ ) an ordinary singularity of multiplicity 2, ( $F \in (I_{2P_1+\dots+2P_6})_5$ ). If there exists a form  $G \neq 0$ ,  $G \in (I_{(4,5)P_1+\dots+(4,5)P_6})_{10}$ , then  $FG \neq 0$  and  $FG \in (I_{(6,7)P_1+\dots+(6,7)P_6})_{15}$ , but this is impossible by the previous case  $r = 1$ .

*Case 4):  $k \not\equiv 2 \pmod{5}$ , and  $d \geq \lceil \frac{12(k+1)}{5} \rceil + 1$ .*

First we will prove that

$$H(\mathcal{Y}, d) = H(X, d) + 12.$$

By Lemma 2.4, it suffices to prove that  $H(\mathcal{Y}, d) = H(X, d) + 12$  for  $d = \lceil \frac{12(k+1)}{5} \rceil + 1 = 12q + \lceil \frac{12r}{5} \rceil + 1$ .

As usual, let  $\nu = 5(k+1) - 2d$ ,  $\mathcal{Y}' = \text{Res}_{\nu\mathcal{C}}\mathcal{Y}$ ,  $X' = \text{Res}_{\nu\mathcal{C}}X$ , and  $d' = d - 12\nu$ . We have:

$r$	$k + 1$	$d$	$\nu$	$k + 1 - 5\nu$	$\mathcal{Y}'$	$X'$	$d'$
0	$5q$	$12q + 1$	$q - 2$	10	$\sum_{i=1}^6 (10, 11)P_i$	$\sum_{i=1}^6 10P_i$	25
1	$5q + 1$	$12q + 4$	$q - 3$	16	$\sum_{i=1}^6 (16, 17)P_i$	$\sum_{i=1}^6 16P_i$	40
2	$5q + 2$	$12q + 6$	$q - 2$	12	$\sum_{i=1}^6 (12, 13)P_i$	$\sum_{i=1}^6 12P_i$	30
4	$5q + 4$	$12q + 11$	$q - 2$	14	$\sum_{i=1}^6 (14, 15)P_i$	$\sum_{i=1}^6 14P_i$	35

Since for  $\nu = 0$ , we have  $\mathcal{Y}' = \mathcal{Y}$ ,  $X' = X$ , and  $d' = d$ , then for every  $\nu \geq 0$  we have:

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'}_{d'}), \quad \dim(I_X)_d = \dim(I_{X'}_{d'}).$$

It follows that

$$H(\mathcal{Y}, d) - H(X, d) = H(\mathcal{Y}', d') - H(X', d').$$

Hence in case  $\nu \geq 0$  we have only to prove that:

- (a)  $H(\sum_{i=1}^6 (10, 11)P_i, 25) = H(\sum_{i=1}^6 10P_i, 25) + 12;$
- (b)  $H(\sum_{i=1}^6 (12, 13)P_i, 30) = H(\sum_{i=1}^6 12P_i, 30) + 12;$
- (c)  $H(\sum_{i=1}^6 (14, 15)P_i, 35) = H(\sum_{i=1}^6 14P_i, 35) + 12;$
- (d)  $H(\sum_{i=1}^6 (16, 17)P_i, 40) = H(\sum_{i=1}^6 16P_i, 40) + 12;$

Now we need the following lemma:

**2.11. Lemma.** *Let:*

$$\mathcal{Y} = (m, m+1)P_1 + \dots + (m, m+1)P_6,$$

$$\tilde{\mathcal{Y}} = (m+2, m+3)P_1 + \dots + (m+2, m+3)P_6,$$

$$\tilde{X} = (m+2)P_1 + \dots + (m+2)P_6.$$

*If the integer  $\eta = 5(d+5) - 12(m+2) + 1 \geq 0$ , and  $H(\mathcal{Y}, d) = \deg \mathcal{Y}$ , then*



- i)  $H(\tilde{\mathcal{Y}}, d+5) = \deg(\tilde{\mathcal{Y}})$ ,  $H(\tilde{X}, d+5) = \deg(\tilde{X})$ ;  
ii)  $H(\tilde{\mathcal{Y}}, d+5) = H(\tilde{X}, d+5) + 12$ .

**Proof.** i) Let  $\mathcal{F}$  be (as above) a rational curve of degree 5 having at each  $P_i$  ( $1 \leq i \leq 6$ ), an ordinary singularity of multiplicity 2. Let  $Q_1, \dots, Q_\eta \in \mathcal{F}$  be generic points. Since  $5(d+5) < 6(2(m+2)) + \eta$ , by Bezout Theorem  $\mathcal{F}$  is a fixed component for the curves defined by the forms of  $(I_{\tilde{\mathcal{Y}}+Q_1+\dots+Q_\eta})_{d+5}$ . It follows that

$$\dim(I_{\tilde{\mathcal{Y}}+Q_1+\dots+Q_\eta})_{d+5} = \dim(I_{\mathcal{Y}})_d.$$

Since  $\binom{d+5+2}{2} - \deg(\tilde{\mathcal{Y}} + Q_1 + \dots + Q_\eta) = \frac{1}{2}(d+7)(d+6) - (\deg \mathcal{Y} + 6(m+2) + 6(m+1) + \eta) = \binom{d+2}{2} - \deg \mathcal{Y} = \binom{d+2}{2} - H(\mathcal{Y}, d) = \dim(I_{\mathcal{Y}})_d$ , we have

$$\dim(I_{\tilde{\mathcal{Y}}+Q_1+\dots+Q_\eta})_{d+5} = \binom{d+5+2}{2} - \deg(\tilde{\mathcal{Y}} + Q_1 + \dots + Q_\eta)$$

hence  $H(\tilde{\mathcal{Y}} + Q_1 + \dots + Q_\eta, d+5) = \deg(\tilde{\mathcal{Y}} + Q_1 + \dots + Q_\eta)$ .

Since obviously  $\tilde{X} \subset \tilde{\mathcal{Y}} \subset \tilde{\mathcal{Y}} + Q_1 + \dots + Q_\eta$ , it follows that  $H(\tilde{\mathcal{Y}}, d+5) = \deg(\tilde{\mathcal{Y}})$ , and  $H(\tilde{X}, d+5) = \deg(\tilde{X})$ .

ii) Obvious. □

By Case 2) we know that  $H(\sum_{i=1}^6 (8, 9)P_i, 20) = H(\sum_{i=1}^6 8P_i, 20) + 12 = \deg(\sum_{i=1}^6 (8, 9)P_i)$ , so by Lemma 2.11 ii) we have (a) :  $H(\sum_{i=1}^6 (10, 11)P_i, 25) = H(\sum_{i=1}^6 10P_i, 25) + 12$ . Moreover, by Lemma 2.11 i),  $H(\sum_{i=1}^6 (10, 11)P_i, 25) = \deg(\sum_{i=1}^6 (10, 11)P_i)$ , hence by Lemma 2.11 ii) we get (b) :  $H(\sum_{i=1}^6 (12, 13)P_i, 30) = H(\sum_{i=1}^6 12P_i, 30) + 12$ . Analogously, by Lemma 2.11, we have that (b)  $\Rightarrow$  (c)  $\Rightarrow$  (d), so, for  $\nu \geq 0$ , we have proved that  $H(\mathcal{Y}, d) = H(X, d) + 12$ .

Now let  $\nu < 0$ . In this case, since  $k+1 = 5q+r \geq 3$ , we are left with the following cases:

$r$	$q$	$k + 1$	$Y$	$X$	$d$
0	1	5	$\sum_{i=1}^6 (5, 6)P_i$	$\sum_{i=1}^6 5P_i$	13
1	1	6	$\sum_{i=1}^6 (6, 7)P_i$	$\sum_{i=1}^6 6P_i$	16
1	2	11	$\sum_{i=1}^6 (11, 12)P_i$	$\sum_{i=1}^6 11P_i$	28
2	1	7	$\sum_{i=1}^6 (7, 8)P_i$	$\sum_{i=1}^6 7P_i$	18
4	0	4	$\sum_{i=1}^6 (4, 5)P_i$	$\sum_{i=1}^6 4P_i$	11
4	1	9	$\sum_{i=1}^6 (9, 10)P_i$	$\sum_{i=1}^6 9P_i$	23

hence we have to prove that:

- (e)  $H(\sum_{i=1}^6 (5, 6)P_i, 13) = H(\sum_{i=1}^6 5P_i, 13) + 12$ ;  
(f)  $H(\sum_{i=1}^6 (6, 7)P_i, 16) = H(\sum_{i=1}^6 6P_i, 16) + 12$ ;  
(g)  $H(\sum_{i=1}^6 (11, 12)P_i, 28) = H(\sum_{i=1}^6 11P_i, 28) + 12$ ;  
(h)  $H(\sum_{i=1}^6 (7, 8)P_i, 18) = H(\sum_{i=1}^6 7P_i, 18) + 12$ ;  
(i)  $H(\sum_{i=1}^6 (4, 5)P_i, 11) = H(\sum_{i=1}^6 4P_i, 11) + 12$ ;

$$(l) \quad H(\sum_{i=1}^6 (9, 10)P_i, 23) = H(\sum_{i=1}^6 9P_i, 23) + 12.$$

By Remark 1.2 and by Lemma 2.10 *iii*), and *iv*), it easily follows that (e) and (i) hold, moreover by Lemma 2.11 we have that (e)  $\Rightarrow$  (h)  $\Rightarrow$  (l)  $\Rightarrow$  (g), and (i)  $\Rightarrow$  (f), so we have proved that  $H(\mathcal{Y}, d) = H(X, d) + 12$  also for  $\nu < 0$ .

Now observe that for  $d = \lceil \frac{12(k+1)}{5} \rceil + 1$ ,  $\dim(I_{\mathcal{Y}})_d$  is positive. In fact, as shown above, we have if  $\nu \geq 0$ :

$$\dim(I_{\mathcal{Y}})_d = \dim(I_{\mathcal{Y}'})_{d'} = \binom{d' + 2}{2} - \deg X' - 12 = \begin{cases} \binom{25+2}{2} - 6\binom{10+1}{2} - 12 = 9 & \text{for } r = 0 \\ \binom{40+2}{2} - 6\binom{16+1}{2} - 12 = 33 & \text{for } r = 1 \\ \binom{30+2}{2} - 6\binom{12+1}{2} - 12 = 16 & \text{for } r = 2 \\ \binom{35+2}{2} - 6\binom{14+1}{2} - 12 = 24 & \text{for } r = 4 \end{cases}$$

if  $\nu < 0$ :

$$\dim(I_{\mathcal{Y}})_d = \begin{cases} \binom{13+2}{2} - 6\binom{5+1}{2} - 12 = 3 & \text{for } r = 0, q = 1 \\ \binom{16+2}{2} - 6\binom{6+1}{2} - 12 = 15 & \text{for } r = 1, q = 1 \\ \binom{28+2}{2} - 6\binom{11+1}{2} - 12 = 27 & \text{for } r = 1, q = 2 \\ \binom{18+2}{2} - 6\binom{7+1}{2} - 12 = 10 & \text{for } r = 2, q = 1 \\ \binom{11+2}{2} - 6\binom{4+1}{2} - 12 = 6 & \text{for } r = 4, q = 0 \\ \binom{23+2}{2} - 6\binom{9+1}{2} - 12 = 18 & \text{for } r = 4, q = 1 \end{cases}.$$

Since  $\dim(I_{\mathcal{Y}})_d > 0$  for  $d = \lceil \frac{12(k+1)}{5} \rceil + 1$ , we have that  $\dim(I_{\mathcal{Y}})_d$  is positive for any  $d \geq \lceil \frac{12(k+1)}{5} \rceil + 1$ , and this means that  $H(\mathcal{Y}, d) < \binom{d+2}{2}$  for any  $d \geq \lceil \frac{12(k+1)}{5} \rceil + 1$ . Moreover, since six generic  $(k+1)$ -fat points impose independent conditions to curves of degree  $d$  if and only if  $d \geq \lfloor \frac{5(k+1)}{2} \rfloor$  (see Remark 1.2), then for  $\lceil \frac{12(k+1)}{5} \rceil + 1 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$ , we have  $H(X, d) < \deg X$ , hence

$$H(\mathcal{Y}, d) = H(X, d) + 12 < \min \left\{ \deg X + 12, \binom{d+2}{2} \right\} = \min \left\{ \deg Y, \binom{d+2}{2} \right\} = \exp H(\mathcal{Y}, d).$$

While for  $d \geq \max \left\{ \lceil \frac{12(k+1)}{5} \rceil + 1; \lfloor \frac{5(k+1)}{2} \rfloor \right\}$ , we have  $H(X, d) = \deg X$ , so  $H(\mathcal{Y}, d) = H(X, d) + 12 = \deg X + 12 = \deg Y$ , and this finish the proof.  $\square$

**2.12. Proposition.** *For  $s = 9$  we have:*

$$H(\mathcal{Y}, d) = \begin{cases} k = 1 : \begin{cases} N + 1 & \text{if } d \leq 8 \\ H(X, d) + 18 = \deg Y & \text{if } d \geq 8 \end{cases} \\ k = 2 : \begin{cases} N + 1 & \text{if } d \leq 10 \\ H(X, d) + 18 = \deg Y & \text{if } d \geq 11 \end{cases} \\ k = 3 : \begin{cases} N + 1 & \text{if } d \leq 13 \\ H(X, d) + 18 = \deg Y & \text{if } d \geq 14 \end{cases} \\ k \geq 4 : \begin{cases} N + 1 & \text{if } d \leq 3k + 3 \\ H(X, d) + 18 = \deg Y & \text{if } d \geq 3k + 4 \end{cases} \end{cases}.$$

**Proof.** For  $k = 1, 2$  the statement is known by [2] and [3].

Let  $k = 3$ , so  $\mathcal{Y} = (4, 5)P_1 + \cdots + (4, 5)P_9$ .

For  $d = 13$ , by CoCoA (see [6]), or by specializing the scheme  $\mathcal{Y}$  it is easy to check that  $\dim(I_{\mathcal{Y}})_{13} = 0$ , hence for  $d \leq 13$  the conclusion follows from Lemma 2.4.

Now let  $C$  be the unique (smooth) cubic curve passing through the support of  $\mathcal{Y}$ , i.e., through  $P_1, \dots, P_9$ . Consider the following exact sequence, where  $\mathcal{Y}' = \text{Res}_C \mathcal{Y}$ :

$$0 \rightarrow \mathcal{I}_{\mathcal{Y}'}(d-3) \rightarrow \mathcal{I}_{\mathcal{Y}}(d) \rightarrow \mathcal{I}_{\mathcal{Y} \cap C, C}(d) \rightarrow 0$$

We have that  $\mathcal{I}_{\mathcal{Y} \cap C, C}(d) = \mathcal{O}_C(dH - \mathcal{Y} \cap C)$ , where  $H$  is a line section of  $C$ , and  $\deg(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 3d - 9(k+1)$ .

Let  $d = 14$ . Since  $k = 3$ , we have  $\deg(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 14 \cdot 3 - 4 \cdot 9 = 6$ . It follows that  $h^1(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 0$ . Since  $\mathcal{Y}' = (3, 4)P_1 + \cdots + (3, 4)P_9$ , from the case  $k = 2$  we get  $h^1(\mathcal{I}_{\mathcal{Y}'}(d-3)) = h^1(\mathcal{I}_{\mathcal{Y}'}(11)) = 0$ . So by the exact sequence above it follows that  $h^1(\mathcal{I}_{(4,5)P_1+\dots+(4,5)P_9}(14)) = 0$ , which implies  $H(\mathcal{Y}, 14) = \deg \mathcal{Y}$ . For  $d > 14$  the conclusion follows from Lemma 2.4.

Let  $k \geq 4$ .

Now we proceed by induction on  $k$ . For  $k = 4$ , we have  $\mathcal{Y} = (5, 6)P_1 + \cdots + (5, 6)P_9$ , and  $3k + 4 = 16$ . By CoCoA (see [6]), or by specializing the scheme  $\mathcal{Y}$  it is easy to check that  $\dim(I_{\mathcal{Y}})_{16} = 0$ . So, since  $N + 1 = \binom{16+2}{2} = 9 \cdot 17 = \deg \mathcal{Y}$ , it follows that  $H(\mathcal{Y}, 16) = N + 1 = \deg \mathcal{Y}$ . Hence by Lemma 2.4 it follows that for  $d \leq 16$  we have  $H(\mathcal{Y}, d) = N + 1$ , while for  $d \geq 16$  we have  $H(\mathcal{Y}, d) = \deg \mathcal{Y}$ .

Now let  $k > 4$ . We have:

$$\mathcal{Y} = (k+1, k+2)P_1 + \cdots + (k+1, k+2)P_9 \quad \mathcal{Y}' = (k, k+1)P_1 + \cdots + (k, k+1)P_9.$$

Since obviously if  $d \leq 3k + 3$ , then  $d - 3 \leq 3(k-1) + 3$ , and if  $d \geq 3k + 4$ , then  $d - 3 \geq 3(k-1) + 4$ , by the induction hypothesis we have  $H(\mathcal{Y}', d-3) = N' + 1$  for  $d - 3 \leq 3(k-1) + 3$ , ( $N' = \binom{d-3+2}{2}$ ), and  $H(\mathcal{Y}', d-3) = \deg \mathcal{Y}'$  for  $d - 3 \geq 3(k-1) + 4$ . That is:

$$h^0(\mathcal{I}_{\mathcal{Y}'}(d-3)) = 0 \quad \text{for } d - 3 \leq 3(k-1) + 3,$$

$$h^1(\mathcal{I}_{\mathcal{Y}'}(d-3)) = 0 \quad \text{for } d - 3 \geq 3(k-1) + 4.$$

Moreover, since  $\deg(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 3d - 9(k+1) \leq 0$  for  $d \leq 3k + 3$ , and  $\deg(\mathcal{O}_C(dH - \mathcal{Y} \cap C)) = 3d - 9(k+1) \geq 3$  for  $d \geq 3k + 4$ , we have:

$$h^0(\mathcal{I}_{\mathcal{Y} \cap C, C}(d)) = 0 \quad \text{for } d \leq 3k + 3,$$

$$h^1(\mathcal{I}_{\mathcal{Y} \cap C, C}(d)) = 0 \quad \text{for } d \geq 3k + 4.$$

So whenever  $d \leq 3k + 3$ , we get  $h^0(\mathcal{I}_{\mathcal{Y}'}(d-3)) = h^0(\mathcal{I}_{\mathcal{Y} \cap C, C}(d)) = 0$ , which by the exact sequence above implies  $h^0(\mathcal{I}_{\mathcal{Y}}(d)) = 0$ .

When  $d \geq 3k + 4$ , we get  $h^1(\mathcal{I}_{\mathcal{Y}'}(d-3)) = h^1(\mathcal{I}_{\mathcal{Y} \cap C, C}(d)) = 0$ , so by the exact sequence above we have  $h^1(\mathcal{I}_{\mathcal{Y}}(d)) = 0$ , and we are done.  $\square$

With all these partial results we have actually proved the main theorem of this paper:

**2.13. Theorem.** *For  $s \leq 6$ , or  $s = 9$ , then*

$$\dim O_{k,d}^s = \min \left\{ H(X, d) + 2s, \binom{d+2}{2} \right\} - 1$$

*except when  $s = 2$ ,  $d = k + 2$  where  $\dim O_{k,k+2}^2 = H(T, d) - 1 = \binom{d+2}{2} - 2$ .*

**Proof.** For  $s = 1$ , since  $H(X, d) = \min \left\{ \binom{k+2}{2}, \binom{d+2}{2} \right\}$ , then the result follows from Remark 1.6.

For  $s = 2$  and  $d = k + 2$ , since  $H(\mathcal{Y}, d) = H(T, d)$  (see Propositions 2.5), by the obvious inequalities  $H(\mathcal{Y}, d) \leq H(Y, d) \leq H(T, d)$  we get

$$H(Y, d) = H(\mathcal{Y}, d) = H(T, d)$$

and the conclusion follows from Remark 1.7 (†).

In the other cases by Lemma 2.2, and from Propositions 2.5 to 2.9, and Proposition 2.12, we have

$$H(Y, d) = H(\mathcal{Y}, d) = \min\{H(X, d) + 2s, N + 1\},$$

hence from Remark 1.7 (†) we get the conclusion.  $\square$

**2.14. Corollary.** *Let  $\delta = \min\{\deg Y - 1, N\} - \dim O_{k,d}^s$  be the defect of  $O_{k,d}^s$ . If  $s \leq 6$ , or  $s = 9$ , then  $O_{k,d}^s$  is defective only in the following cases:*

- i)  $s = 2$ ,  $d = k + 2$ , with defect:  $\delta = 1$ .
- ii)  $s = 2$ ,  $k \geq 3$ ,  $k + 3 \leq d \leq 2k$ , with defect:  $\delta = \min \left\{ \binom{2(k+1)-d}{2}; (d-k)^2 - 4 \right\}$ .
- iii)  $s = 3$ ,  $k \geq 7$ ,  $k$  odd,  $\lceil \frac{3(k+1)}{2} \rceil + 2 \leq d \leq 2k$ , with defect:  $\delta = \min \left\{ 3 \binom{2(k+1)-d}{2}; \binom{2d-3k-1}{2} - 6 \right\}$ .
- iv)  $s = 3$ ,  $k \geq 6$ ,  $k$  even,  $\lceil \frac{3(k+1)}{2} \rceil + 1 \leq d \leq 2k$ , with defect:  $\delta = \min \left\{ 3 \binom{2(k+1)-d}{2}; \binom{2d-3k-1}{2} - 6 \right\}$ .
- v)  $s = 5$ ,  $k \geq 5$ ,  $2k + 4 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$ , with defect  $\delta = \min \left\{ \binom{5(k+1)-2d}{2}; 5 \binom{d-2k-1}{2} - 9 \right\}$ .
- vi)  $s = 6$ ,  $k \equiv 2 \pmod{5}$ ,  $k \geq 17$ ,  $\lceil \frac{12(k+1)}{5} \rceil \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$ , with defect:  

$$\delta = \min \left\{ 6 \binom{5(k+1)-2d}{2}; \binom{5d-12k-10}{2} - 12 \right\}.$$
- vii)  $s = 6$ ,  $k \not\equiv 2 \pmod{5}$ ,  $k \geq \begin{cases} 19 & \text{if } k \text{ odd} \\ 24 & \text{if } k \text{ even} \end{cases}$ ,  $\lceil \frac{12(k+1)}{5} \rceil + 1 \leq d \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$ , with defect:  

$$\delta = \min \left\{ 6 \binom{5(k+1)-2d}{2}; \binom{5d-12k-10}{2} - 12 \right\}.$$

**Proof.** First observe that:  $k + 3 \leq 2k$  implies  $k \geq 3$ ; if  $k$  is odd and  $\lceil \frac{3(k+1)}{2} \rceil + 2 \leq 2k$ , then  $3(k+1) + 4 \leq 4k$ , that is  $k \geq 7$ , while if  $k$  is even and  $\lceil \frac{3(k+1)}{2} \rceil + 1 \leq 2k$ , then  $k \geq 6$ ; from  $2k + 4 \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$  we get  $k \geq 5$ ; finally, for  $k \equiv 2 \pmod{5}$ , it is easy to compute that  $\lceil \frac{12(k+1)}{5} \rceil \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$  implies  $k \geq 17$ , while for  $k \not\equiv 2 \pmod{5}$ , if  $\lceil \frac{12(k+1)}{5} \rceil + 1 \leq \lfloor \frac{5(k+1)}{2} \rfloor - 1$ , then  $k \geq \begin{cases} 19 & \text{if } k \text{ odd} \\ 24 & \text{if } k \text{ even} \end{cases}$ .

From what we have seen above, and by Remark 1.6, Propositions 2.5 to 2.9 and 2.12, we immediately get that  $O_{k,d}^s$  is defective only in the cases i) to vii).

For  $s = 2$  and  $d = k + 2$ , since  $\dim O_{k,k+2}^2 = N - 1$ , while the expected dimension is  $N$ , we have  $\delta = 1$ . In the other cases we know that  $H(Y, d) = H(X, d) + 2s$ , so we have

$$\begin{aligned}\delta &= \min\{\deg Y - 1, N\} - \dim O_{k,d}^s = \min\{\deg Y - 1, N\} - H(Y, d) + 1 \\ &= \min\{\deg Y - H(X, d) - 2s, N + 1 - H(X, d) - 2s\} = \min\{\deg X - H(X, d), \dim(I_X)_d - 2s\}.\end{aligned}$$

For  $s = 2$ ,  $k \geq 3$  and  $k + 3 \leq d \leq 2k$ , computing the dimension of  $(I_X)_d$  by removing the line  $P_1P_2$   $(2(k + 1) - d)$  times, we get:

$$\dim(I_X)_d = \dim(I_{X'})_{2(d-k-1)} = \binom{2(d-k-1)+2}{2} - 2\binom{d-k}{2} = (d-k)^2,$$

where  $X' = (d-k-1)P_1 + (d-k-1)P_2$ , hence

$$\deg X - H(X, d) = 2\binom{k+2}{2} - \binom{d+2}{2} + (d-k)^2 = \binom{2(k+1)-d}{2},$$

$$\delta = \min\left\{\binom{2(k+1)-d}{2}; (d-k)^2 - 4\right\}.$$

In cases *iii*) and *iv*), computing the dimension of  $(I_X)_d$  by cutting off the three lines  $P_1P_2$ ,  $P_1P_3$ ,  $P_2P_3$ ,  $2(k + 1) - d$  times each, we have:

$$\begin{aligned}\dim(I_X)_d &= \dim(I_{X'})_{d-3(2k+2-d)} = \dim(I_{X'})_{2(2d-3k-3)} \\ &= \binom{2(2d-3k-3)+2}{2} - 3\binom{2d-3k-2}{2} = \binom{2d-3k-1}{2},\end{aligned}$$

where  $X' = \sum_{i=1}^3 (k+1-2(2k+2-d))P_i = \sum_{i=1}^3 (2d-3k-3)P_i$ , and from here we easily get:

$$\deg X - H(X, d) = 3\binom{k+2}{2} - \binom{d+2}{2} + \binom{2d-3k-1}{2} = 3\binom{2(k+1)-d}{2},$$

$$\delta = \min\left\{3\binom{2(k+1)-d}{2}; \binom{2d-3k-1}{2} - 6\right\}.$$

For  $s=5$ , computing the dimension of  $(I_X)_d$  (by cutting off the fixed conics), we get:

$$\begin{aligned}\dim(I_X)_d &= \dim(I_{X'})_{d-2(5k+5-2d)} = \dim(I_{X'})_{5(d-2k-2)} \\ &= \binom{5(d-2k-2)+2}{2} - 5\binom{2d-4k-3}{2} = 5\binom{d-2k-1}{2} + 1,\end{aligned}$$

where  $X' = \sum_{i=1}^5 (k+1-(5k+5-2d))P_i = \sum_{i=1}^5 (2d-4k-4)P_i$ , and from here we get:

$$\deg X - H(X, d) = 5\binom{k+2}{2} - \binom{d+2}{2} + 5\binom{d-2k-1}{2} + 1 = \binom{5(k+1)-2d}{2},$$

$$\delta = \min\left\{\binom{5(k+1)-2d}{2}; 5\binom{d-2k-1}{2} - 9\right\}.$$

Finally, for  $s=6$ , calculating the dimension of  $(I_X)_d$  by removing every conic  $C_i$  (see the proof of Proposition 2.9)  $(5(k+1) - 2d)$  times, we get

$$\begin{aligned} \dim(I_X)_d &= \dim(I_{X'})_{d-12(5k+5-2d)} = \dim(I_{X'})_{25d-60k-60} \\ &= \binom{25d-60k-60+2}{2} - 6 \binom{10d-24k-24+1}{2} = \binom{5d-12k-10}{2}, \end{aligned}$$

where  $X' = \sum_{i=1}^6 (k+1 - 5(5k+5-2d))P_i = \sum_{i=1}^6 (10d-24k-24)P_i$ , and from here we get:

$$\deg X - H(X, d) = 6 \binom{k+2}{2} - \binom{d+2}{2} + \binom{5d-12k-10}{2} = 6 \binom{5(k+1)-2d}{2},$$

$$\delta = \min \left\{ 6 \binom{5(k+1)-2d}{2}; \binom{5d-12k-10}{2} - 12 \right\}.$$

□

**2.15. Remark.** Some examples, some computations, and a lack of geometric reasons, lead us to conjecture that also for  $s = 7$ , and  $s = 8$  we have

$$\dim O_{k,d}^s = \min \left\{ H(X, d) + 2s, \binom{d+2}{2} \right\} - 1.$$

Unfortunately, by methods similar to the ones utilized for  $s \leq 6$ , the proof splits into many cases, and becomes too long and tedious to justify including.

E.Ballico and C.Fontanari in [4] give partial results about the regularity of  $O_{k,d}^s$  for  $2 \leq s \leq 8$ . Our Corollary 2.14, for  $s \leq 6$  or  $s = 9$ , improves the results of [4] and gives a complete classification of all the defective cases.

**2.16. Remark.** We wish to notice that there are no defective cases for  $s = 4$  or  $s = 9$ .

In case  $s = 2$ ,  $d = k + 2$  defectivity is forced by the defectivity of  $T$ , in fact, since  $Y \subset T$  implies that  $H(Y, k+2) \leq H(T, k+2)$ , and since  $H(T, k+2) = N < \exp H(Y, k+2) = N + 1$ , it follows that  $H(Y, k+2) < \exp H(Y, k+2)$ . In the other cases defectivity of  $O_{k,d}^s$  is forced by the defectivity of  $X$ .

**2.17. Remark.** In light of Remarks 2.15 and 2.16, and the results of L.Evain (see Remark 1.2), we like to conjecture that if  $s$  is a square, then  $O_{k,d}^s$  is regular in any degree  $d$ .

Anyway by the results of L.Evain, and by [5], Lemma 3.1, we easily deduce a partial result about the regularity of  $O_{k,d}^s$ :

*If  $s$  is a square, and  $N + 1 \leq \deg X$  or  $N + 1 \geq \deg T$ , then  $\dim O_{k,d}^s$  is as expected.*

In fact if  $s$  is a square, by [9] we know that  $X$  and  $T$  have maximal Hilbert function. Hence if  $N + 1 \leq \deg X$ , then  $\dim(I_X)_d = 0$ , and if  $N + 1 \geq \deg T$ , then  $H(T, d) = \deg T$ . Since  $X \subset Y \subset T$ , it follows that if  $\dim(I_X)_d = 0$ , then  $H(Y, d) = N + 1$ , and if  $H(T, d) = \deg T$ , then  $H(Y, d) = \deg Y$ , and now the conclusion follows from Remark 2.1.

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